Available online at www.sciencedirect.com

SCIENCE DIRECT®

International Journal of Solids and Structures 41 (2004) 7459–7469

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

A new two-point deformation tensor and its relation to the classical kinematical framework and the stress concept

Nikolaus Böck, Gerhard A. Holzapfel *

Institute for Structural Analysis—Computational Biomechanics, Graz University of Technology, Schiesstattgasse 14-B, A–8010 Graz, Austria

Received 5 July 2003; received in revised form 7 June 2004
Available online 4 August 2004

Abstract

Starting from the issue of what is the correct form for a Legendre transformation of the strain energy in terms of Eulerian and two-point tensor variables we introduce a *new* two-point deformation tensor, namely $\mathbf{H} = (\mathbf{F} - \mathbf{F}^{-\mathrm{T}})/2$, as a possible deformation measure involving points in two distinct configurations. The Lie derivative of $\mathbf{H}$ is work conjugate to the first Piola–Kirchhoff stress tensor $\mathbf{P}$. The deformation measure $\mathbf{H}$ leads to straightforward manipulations within a two-point setting such as the derivation of the virtual work equation and its linearization required for finite element implementation. The manipulations are analogous to those used for the Lagrangian and Eulerian frameworks. It is also shown that the Legendre transformation in terms of two-point tensors and spatial tensors require Lie derivatives. As an illustrative example we propose a simple Saint Venant–Kirchhoff type of a strain-energy function in terms of $\mathbf{H}$. The constitutive model leads to physically meaningful results also for the large compressive strain domain, which is not the case for the classical Saint Venant–Kirchhoff material.

Keywords: Nonlinear continuum mechanics; Finite element method; Legendre transformation; Two-point tensor; Lie derivative; Constitutive equation

1. Introduction

For the rate $\dot{\Psi}$ of the free Helmholtz-energy function Ψ it is common practice to use the expressions $\mathbf{S} : \dot{\mathbf{E}}$, where $\mathbf{S}$ is the second Piola–Kirchhoff stress tensor and $\dot{\mathbf{E}}$ the material time derivative of the Green–Lagrange strain tensor, or $\boldsymbol{\tau} : \mathbf{d}$, where $\boldsymbol{\tau}$ is the Kirchhoff stress tensor and $\mathbf{d}$ the rate of deformation tensor, or $\mathbf{P} : \dot{\mathbf{F}}$, where $\mathbf{P}$ is the first Piola–Kirchhoff stress tensor and $\dot{\mathbf{F}}$ the material time derivative of the deformation gradient (both are two-point tensors). By dealing with phase transformations it is often more useful to employ the Gibbs free energy g, which may be obtained from Ψ by means of a Legendre transformation. While it is obvious that the Legendre transformation $g = \Psi - \mathbf{S} : \mathbf{E}$ in terms of material

* Corresponding author. Tel.: +43-3168731625; fax: +43-3168731615.
E-mail address: gh@biomech.tu-graz.ac.at (G.A. Holzapfel).
URL: http://www.cis.tu-graz.ac.at/biomech.

doi:10.1016/j.ijsolstr.2004.06.008

tensors leads to $\dot{g} = -\mathbf{E} : \dot{\mathbf{S}}$, the situation in an Eulerian description or a two-point formulation is not so obvious. Note that, for example, $\mathbf{E} : \dot{\mathbf{S}} \neq \mathbf{F} : \dot{\mathbf{P}}$.

One aim of the paper is to show that it is possible to derive the Gibbs free energy in terms of Eulerian or two-point tensors by the use of well-known algebraic manipulations. Consequently, some interesting new results occur. In Section 2 we introduce a new two-point deformation tensor $\mathbf{H}$ and discuss material and Lie derivatives as well as associated properties. In addition, we introduce some necessary definition of strain and stress measures. Section 3 starts by introducing the Legendre transformation in terms of material tensors—associated properties are discussed. Legendre transformations in Eulerian and two-point forms are derived, where Lie derivatives are required. Section 4 aims to discuss the relationships between the stress power in terms of $\mathbf{P}$, $\mathbf{F}$ and $\mathbf{H}$. In Section 5 we introduce the internal virtual work and its linearization in terms of the two-point tensors $\mathbf{P}$ and $\mathbf{H}$, which is the departure for the finite element method. The elasticity tensor is provided. We show that the associated structure is similar to that of the well-known results obtained from the Lagrangian and Eulerian descriptions, and summarize the results in a table. As an illustrative example in Section 6 we propose a simple Saint Venant–Kirchhoff type of a strain-energy function in terms of $\mathbf{H}$ and derive the constitutive relation $\mathbf{P} = \mathbf{P}(\mathbf{H})$. Based on the properties of $\mathbf{H}$ we obtain physically meaningful results also for the large strain domain, which is not the case for the classical Saint Venant–Kirchhoff material. Appendix A provides a general form for $\mathbf{P} = \mathbf{P}(\mathbf{H})$, which is derived from the free Helmholtz-energy function.

2. Definition of kinematical quantities and stress tensors

In this section we introduce some necessary definition of deformation, strain and stress measures in addition to relevant Lie derivatives.

We consider a reference frame of coordinate axes at a fixed origin with orthonormal basis vectors and introduce a *new* (covariant) deformation tensor of the form

$$\mathbf{H} = \frac{1}{2}(\mathbf{F} - \mathbf{F}^{-\mathrm{T}}). \tag{1}$$

Herein $\mathbf{F}$ is the deformation gradient, with $\det\mathbf{F} = J > 0$, where J denotes the volume ratio. The two-point tensor $\mathbf{H}$ may be seen as the transformation $\mathbf{H} = \mathbf{F}^{-\mathrm{T}}\mathbf{E}$ (or $\mathbf{H} = \mathbf{eF}$), where $\mathbf{E}$ is the Green–Lagrange strain tensor and $\mathbf{e} = \mathbf{F}^{-\mathrm{T}}\mathbf{E}\mathbf{F}^{-1}$ the Euler–Almansi strain tensor. As can be seen, $\mathbf{H}$ follows from $\mathbf{E}$ by transforming one fixed set of three basis vectors to the current configuration, while $\mathbf{e}$ follows from $\mathbf{E}$ by a classical push-forward operation, i.e. $\mathbf{E} \to \mathbf{H} \to \mathbf{e}$. Note that for $\mathbf{F} = \mathbf{I}$ we get $\mathbf{H} = \mathbf{O}$.

To get an idea about the physical interpretation of $\mathbf{H}$ we consider a one dimensional problem, for example, a rod, which is elongated from the initial length l_0 up to the current length l. Thus, we have

$$E = \frac{1}{2}\frac{l^2 - l_0^2}{l_0^2} = \frac{1}{2}(\lambda^2 - 1), \tag{2}$$

$$H = \frac{1}{2}\frac{l^2 - l_0^2}{l l_0} = \frac{1}{2}\left(\lambda - \frac{1}{\lambda}\right), \tag{3}$$

$$e = \frac{1}{2}\frac{l^2 - l_0^2}{l^2} = \frac{1}{2}\left(1 - \frac{1}{\lambda^2}\right), \tag{4}$$

where $\lambda = l/l_0$ denotes the stretch ratio. Fig. 1 illustrates the distribution of the different measures E, H and e along λ varying between 0^+ and $+\infty$. An interesting feature of H is the fact that the deformation measure goes to $+\infty$ if l goes to plus infinity, and the deformation measure goes to $-\infty$ if l goes to zero. However,

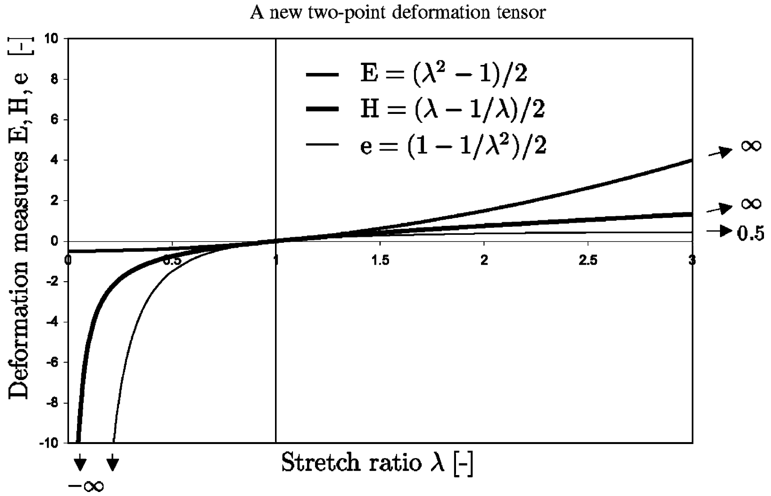

Fig. 1. Deformation measures E, H and e versus the stretch ratio $\lambda = l/l_0$ showing the tendency of $H \to +\infty$ as $\lambda \to +\infty$ and $H \to -\infty$ as $\lambda \to 0$. Note that $E = -1/2$ for $\lambda \to 0$ and $e = 1/2$ for $\lambda \to +\infty$.

as well-known, for the case of E and e, we have the situation that if l goes to $+\infty$, then e goes to 1/2, and if l goes to zero then E goes to $-1/2$, which are physical awkward results.

In order to find an expression in which the two-point tensor **H** is given in terms of principal directions and principal stretches we start from the relation

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \sum_{a=1}^{3} \lambda_a (\mathbf{R}\widehat{\mathbf{N}}_a) \otimes \widehat{\mathbf{N}}_a = \sum_{a=1}^{3} \lambda_a \hat{\mathbf{n}}_a \otimes \widehat{\mathbf{N}}_a \tag{5}$$

for the deformation gradient **F**, where the spectral decomposition $\mathbf{U} = \sum_{a=1}^{3} \lambda_a \widehat{\mathbf{N}}_a \otimes \widehat{\mathbf{N}}_a$ for the (symmetric) right stretch tensor **U** has been used. Herein, $\widehat{\mathbf{N}}_a$, $a = 1, 2, 3$, are the principal referential directions (mutually orthogonal and normalized eigenvectors of the material tensor **U**), λ_a are their corresponding principal stretches (eigenvalues of **U**), and $\hat{\mathbf{n}}_a = \mathbf{R}\widehat{\mathbf{N}}_a$ are the corresponding principal spatial directions, where **R** is the (proper orthogonal) rotation tensor. Consequently, we may write

$$\mathbf{F}^{-\mathrm{T}} = \mathbf{R}\mathbf{U}^{-1} = \sum_{a=1}^{3} \lambda_a^{-1} \hat{\mathbf{n}}_a \otimes \widehat{\mathbf{N}}_a, \tag{6}$$

where the property $\mathbf{R}^{\mathrm{T}}\mathbf{R} = \mathbf{I}$ for the orthogonal tensor **R** has been used. Hence, by means of $(5)_3$ and $(6)_2$, **H** may be expressed in terms of principal stretches and principal directions. From definition (1) we finally get the useful relationship

$$\mathbf{H} = \frac{1}{2} \sum_{a=1}^{3} (\lambda_a - \lambda_a^{-1}) \hat{\mathbf{n}}_a \otimes \widehat{\mathbf{N}}_a. \tag{7}$$

For the sake of completeness we express the deformation tensor **H** as the gradient of the displacement field **u** of a typical particle. By the use of Grad(•) and grad(•) for the derivatives of (•) with respect to the reference and current positions we have $\mathbf{F} = \mathbf{I} + \mathrm{Grad}\mathbf{u}$ and $\mathbf{F}^{-1} = \mathbf{I} - \mathrm{grad}\mathbf{u}$. Thus, we may write Eq. (1) in the form

$$\mathbf{H} = \frac{1}{2}[\mathrm{Grad}\mathbf{u} + (\mathrm{grad}\mathbf{u})^{\mathrm{T}}]. \tag{8}$$

The Lie derivatives of $\mathbf{H}$ and $\mathbf{e}$, denoted by $\pounds_{\mathbf{v}}\mathbf{H}$ and $\pounds_{\mathbf{v}}\mathbf{e}$, follow with the concept of directional derivatives (see, for example, Holzapfel, 2000). Thus, we have

$$\pounds_{\mathbf{v}}\mathbf{H} = \mathbf{F}^{-\mathrm{T}}\dot{\mathbf{E}} = \mathbf{F}^{-\mathrm{T}}(\overline{\mathbf{F}^{\mathrm{T}}\mathbf{H}})^{\cdot}, \tag{9}$$

$$\pounds_{\mathbf{v}}\mathbf{e} = \mathbf{F}^{-\mathrm{T}}\dot{\mathbf{E}}\mathbf{F}^{-1} = \mathbf{F}^{-\mathrm{T}}(\overline{\mathbf{F}^{\mathrm{T}}\mathbf{e}\mathbf{F}})^{\cdot}\mathbf{F}^{-1}, \tag{10}$$

where the material time derivative is denoted by a superposed dot. Clearly, from relation $(9)_2$ we find the material time derivative of $\mathbf{H}$ to be

$$\dot{\mathbf{H}} = -\mathbf{l}^{\mathrm{T}}\mathbf{H} + \pounds_{\mathbf{v}}\mathbf{H}, \tag{11}$$

where $\mathbf{l} = \dot{\mathbf{F}}\mathbf{F}^{-1}$ denotes the spatial velocity gradient.

Regarding the concept of stress we introduce the two-point tensor $\mathbf{P}$, i.e. the first Piola–Kirchhoff stress tensor, by analogy with the above procedure. It may be seen as the transformation $\mathbf{P} = \mathbf{FS}$ (or $\mathbf{P} = \boldsymbol{\tau}\mathbf{F}^{-\mathrm{T}}$), where $\mathbf{S}$ is the second Piola–Kirchhoff stress tensor and $\boldsymbol{\tau} = \mathbf{FSF}^{\mathrm{T}}$ denotes the Kirchhoff stress tensor ($\mathbf{S} \to \mathbf{P} \to \boldsymbol{\tau}$).

The Lie derivatives of $\mathbf{P}$ and $\boldsymbol{\tau}$ are

$$\pounds_{\mathbf{v}}\mathbf{P} = \mathbf{F}\dot{\mathbf{S}} = \mathbf{F}(\overline{\mathbf{F}^{-1}\mathbf{P}})^{\cdot}, \tag{12}$$

$$\pounds_{\mathbf{v}}\boldsymbol{\tau} = \mathbf{F}\dot{\mathbf{S}}\mathbf{F}^{\mathrm{T}} = \mathbf{F}(\overline{\mathbf{F}^{-1}\boldsymbol{\tau}\mathbf{F}^{-\mathrm{T}}})^{\cdot}\mathbf{F}^{\mathrm{T}}, \tag{13}$$

which is the analogue of Eqs. (9) and (10).

3. Legendre transformation

In this section we aim to derive Legendre transformations, in particular, in terms of $\mathbf{H}$, $\mathbf{P}$ and $\mathbf{e}$, $\boldsymbol{\tau}$.

We consider hyperelastic materials within a purely mechanical framework. We introduce a free Helmholtz-energy function $\Psi = \Psi(\mathbf{E})$, defined per unit volume, in which Ψ is solely a function of $\mathbf{E}$ such that the physical expression of the form

$$\mathbf{S} = \frac{\partial \Psi(\mathbf{E})}{\partial \mathbf{E}} \tag{14}$$

holds. Thus, the rate of Ψ, which in our case is the stress power $\dot{\Psi}$, is given by

$$\dot{\Psi}(\mathbf{E}) = \mathbf{S} : \dot{\mathbf{E}}, \tag{15}$$

where $\mathbf{S}$ and $\mathbf{E}$ are work conjugate variables. Herein $\mathbf{E}$ is the independent state variable and $\mathbf{S} = \mathbf{S}(\mathbf{E})$ is the dependent state variable. They are interrelated via the constitutive relation (14).

Now we introduce a new function $g = g(\mathbf{S})$, depending on the independent variable $\mathbf{S}$, which requires invertibility of (14), at least locally. Hence, we use the transformation of the form (see, Courant and Hilbert, 1962, pp. 32–39)

$$g(\mathbf{S}) = \Psi(\mathbf{E}(\mathbf{S})) - \mathbf{S} : \mathbf{E}(\mathbf{S}), \tag{16}$$

which is called Legendre transformation, $\Psi \to g$. Thus, by computing the material time derivative and using Eq. (15) we get

$$\dot{g}(\mathbf{S}) = \dot{\Psi}(\mathbf{E}) - \mathbf{S} : \dot{\mathbf{E}} - \mathbf{E} : \dot{\mathbf{S}} = -\mathbf{E} : \dot{\mathbf{S}}. \tag{17}$$

Before proceeding it seems to be beneficial to briefly explain some characteristics of the Legendre transformation (Gyarmati, 1970; Callen, 1985; Šilhavý, 1997).

By using $\Psi = \Psi(\mathbf{E})$ and Eq. (14) we are able to eliminate $\mathbf{E}$ in favour of $\mathbf{S}$. However, if we are using the transformation $\hat{\Psi} = \hat{\Psi}[\mathbf{S}(\mathbf{E})]$ instead of Eq. (16) then we are not preserving the full information because, for this case, $\mathbf{S}$ only contains information about the gradient of Ψ. As Callen (1985, Section 5.2) pointed out we can only describe the shape of a function if we know its gradient in each point, but there still remains unknown constants in order to fully describe the function.

At this point we provide a brief geometrical interpretation of the Legendre transformation. Given a function $\psi = \psi(E)$, which depends on the variable E, and hence $S(E) = \mathrm{d}\psi/\mathrm{d}E$ is the gradient of the function in each point. If we desire to consider S as an independent variable in place of E, then we might find E as a function of S by eliminating E between the above mentioned relations. However, thereby we would lose some of the information of the given relation $\psi = \psi(E)$, because knowing the gradient of ψ in each point enables us only to restore the shape of the function (the shape of the curve in our example), and not its absolute spatial position. Additionally, if we know the intersection of each tangent of our curve with the abscissa, we will also know the absolute spatial position of the curve. By calling these intersections g, we may write it as a function of the gradients, i.e. $g = g(S)$. The function g now contains the identical information as function ψ. From the geometrical point of view this means that we can describe a given curve equally well either as the locus of points satisfying the relation $\psi = \psi(E)$ or as the envelope of a family of tangent lines. It is obvious from our example that we can get g via the relation $g = \psi - SE$. This transformation, which stores the full information about the curve, is a Legendre transformation. Hence, we see that we may derive the variable E by means of $E(S) = -\mathrm{d}g/\mathrm{d}S$, and again we may derive a Legendre transformation $\psi = g + SE$.

Having this in mind, we continue by applying the Legendre transformation to the Gibbs free energy g. Starting from (16) it follows that $\Psi = g + \mathbf{S} : \mathbf{E}$, and $\mathbf{E} = -\partial g(\mathbf{S})/\partial \mathbf{S}$. Thus, material time derivative gives with $(17)_2$

$$\dot{\Psi} = \dot{g} + \mathbf{S} : \dot{\mathbf{E}} + \dot{\mathbf{S}} : \mathbf{E} = \mathbf{S} : \dot{\mathbf{E}} \tag{18}$$

(compare with (15)).

In an analogous manner we consider now Legendre transformations in terms of $\mathbf{H}$, $\mathbf{P}$ and $\mathbf{e}$, $\boldsymbol{\tau}$. Straightforward tensor manipulations give

$$\mathbf{S} : \mathbf{E} = \mathbf{F}\mathbf{S} : \mathbf{F}^{-\mathrm{T}}\mathbf{E} = \mathbf{P} : \mathbf{H}, \tag{19}$$

and

$$\mathbf{S} : \mathbf{E} = \mathbf{F}\mathbf{S}\mathbf{F}^{\mathrm{T}} : \mathbf{F}^{-\mathrm{T}}\mathbf{E}\mathbf{F}^{-1} = \boldsymbol{\tau} : \mathbf{e}. \tag{20}$$

From Eqs. (19) and (20) it is attempting to postulate that $\mathbf{H}$ and $\mathbf{e}$ are the work conjugate variables to $\mathbf{P}$ and $\boldsymbol{\tau}$. However, as can be shown, the relations

$$\boldsymbol{\tau} \neq \frac{\partial \Psi}{\partial \mathbf{e}} \quad \text{and} \quad \mathbf{P} \neq \frac{\partial \Psi}{\partial \mathbf{H}} \tag{21}$$

hold (isotropy has been assumed). Hence, it seems not to be possible to define a new variable in the sense of Eq. (14) and Legendre transformations of the forms $\Psi(\mathbf{e}) \to g(\boldsymbol{\tau})$ and $\Psi(\mathbf{H}) \to g(\mathbf{P})$. However, as shown in the following, it is possible to define Legendre transformations by means of push-forward operations and Lie derivatives. Starting from (14) and (15), we obtain by means of $(9)_1$ the relation

$$\dot{\Psi} = \mathbf{F}\frac{\partial \Psi}{\partial \mathbf{E}} : \mathbf{F}^{-\mathrm{T}}\dot{\mathbf{E}} = \mathbf{F}\frac{\partial \Psi}{\partial \mathbf{E}} : \pounds_{\mathbf{v}}\mathbf{H}. \tag{22}$$

If we express the free Helmholtz energy in terms of two-point tensors, then, from Eq. (22) it follows that $\mathbf{H}$ is the independent state variable and $\mathbf{F}(\partial\Psi/\partial\mathbf{E})$ is the dependent variable, where $\mathbf{F}(\partial\Psi/\partial\mathbf{E}) = \mathbf{FS} = \mathbf{P}$. Hence, the analogue of Eq. (15) in a two-point formulation gives the stress power

$$\dot{\Psi} = \mathbf{P} : \pounds_v\mathbf{H}. \tag{23}$$

By means of Eq. $(10)_1$ we find the associated relation in the current configuration

$$\dot{\Psi} = \mathbf{F}\frac{\partial\Psi}{\partial\mathbf{E}}\mathbf{F}^{\mathrm{T}} : \mathbf{F}^{-\mathrm{T}}\dot{\mathbf{E}}\mathbf{F}^{-1} = \mathbf{F}\frac{\partial\Psi}{\partial\mathbf{E}}\mathbf{F}^{\mathrm{T}} : \pounds_v\mathbf{e}. \tag{24}$$

Herein $\mathbf{e}$ is the independent state variable and $\mathbf{F}(\partial\Psi/\partial\mathbf{E})\mathbf{F}^{\mathrm{T}}$ is the dependent variable, where $\mathbf{F}(\partial\Psi/\partial\mathbf{E})\mathbf{F}^{\mathrm{T}} = \mathbf{FSF}^{\mathrm{T}} = \boldsymbol{\tau}$. Hence, the analogue of Eq. (15) in terms of spatial tensors reads

$$\dot{\Psi} = \boldsymbol{\tau} : \pounds_v\mathbf{e}. \tag{25}$$

We are now able to define the Legendre transformation $g = \Psi - \mathbf{P} : \mathbf{H}$ in terms of two-point tensors. Because of the fact that Ψ is a scalar it is straightforward to show that the Lie derivative of the two-point tensors $\mathbf{P}$ and $\mathbf{H}$ is equal to $\dot{\overline{\mathbf{P} : \mathbf{H}}}$. Hence, with this property and Eq. (23) we get

$$\dot{g} = \dot{\Psi} - \mathbf{P} : \pounds_v\mathbf{H} - \pounds_v\mathbf{P} : \mathbf{H} = -\mathbf{H} : \pounds_v\mathbf{P}, \tag{26}$$

and hence, with $\mathbf{E} = -\partial g(\mathbf{S})/\partial\mathbf{S}$ and $\mathbf{H} = \mathbf{F}^{-\mathrm{T}}\mathbf{E}$, we obtain

$$\dot{g} = \mathbf{F}^{-\mathrm{T}}\frac{\partial g}{\partial\mathbf{S}} : \pounds_v\mathbf{P}. \tag{27}$$

The Legendre transformation $g = \Psi - \boldsymbol{\tau} : \mathbf{e}$ in terms of spatial tensors follows by means of $\dot{\overline{\boldsymbol{\tau} : \mathbf{e}}} = \pounds_v(\boldsymbol{\tau} : \mathbf{e})$, and Eq. (25) to

$$\dot{g} = \dot{\Psi} - \boldsymbol{\tau} : \pounds_v\mathbf{e} - \pounds_v\boldsymbol{\tau} : \mathbf{e} = -\mathbf{e} : \pounds_v\boldsymbol{\tau}, \tag{28}$$

with $\mathbf{e} = -\mathbf{F}^{-\mathrm{T}}[\partial g(\mathbf{S})/\partial\mathbf{S}]\mathbf{F}^{-1}$, we obtain

$$\dot{g} = \mathbf{F}^{-\mathrm{T}}\frac{\partial g}{\partial\mathbf{S}}\mathbf{F}^{-1} : \pounds_v\boldsymbol{\tau}. \tag{29}$$

The objective time derivative, the variation and the linearization of a spatial tensor (or a two-point tensor) are based on the concept of Lie derivatives, and hence this concept is also applicable for the Legendre transformation of a spatial tensor (or a two-point tensor).

4. Properties of the stress power $\mathbf{P} : \pounds_v\mathbf{H}$

In this section we aim to discuss the relations between the stress power $\mathbf{P} : \pounds_v\mathbf{H}$ and the stress power $\mathbf{P} : \dot{\mathbf{F}}$.

As we know from the literature, one form of the stress power is $\dot{\Psi} = \mathbf{P} : \dot{\mathbf{F}}$, which in combination with (23), results to $\dot{\Psi} = \mathbf{P} : \pounds_v\mathbf{H} = \mathbf{P} : \dot{\mathbf{F}}$. Note that, in general, $\dot{\mathbf{F}} \neq \pounds_v\mathbf{H}$, as can be seen from (9).

By means of Eqs. (23) and $(9)_1$ we find that

$$\dot{\Psi} = \mathbf{P} : \pounds_v\mathbf{H} = \mathbf{P} : \mathbf{F}^{-\mathrm{T}}\mathrm{sym}(\mathbf{F}^{\mathrm{T}}\dot{\mathbf{F}}) = \mathbf{P} : \frac{1}{2}(\dot{\mathbf{F}} + \mathbf{F}^{-\mathrm{T}}\dot{\overline{\mathbf{F}^{\mathrm{T}}}}\mathbf{F}). \tag{30}$$

In addition, we find by means of the symmetric tensor $\mathbf{S} = \mathbf{F}^{-1}\mathbf{P}$ that

$$\dot{\Psi} = \mathbf{P} : \pounds_v\mathbf{H} = \mathbf{F}^{-1}\mathbf{P} : \mathrm{sym}(\mathbf{F}^{\mathrm{T}}\dot{\mathbf{F}}) = \mathbf{S} : \dot{\overline{\mathbf{F}^{\mathrm{T}}}}\mathbf{F} = \mathbf{P} : \mathbf{F}^{-\mathrm{T}}\dot{\overline{\mathbf{F}^{\mathrm{T}}}}\mathbf{F}. \tag{31}$$

In summary, we achieve as an extension of Eq. (23) the relationships

$$\dot{\Psi} = \mathbf{P} : \pounds_{\mathbf{v}}\mathbf{H} = \mathbf{P} : \frac{1}{2}(\dot{\mathbf{F}} + \mathbf{F}^{-\mathrm{T}}\dot{\overline{\mathbf{F}^{\mathrm{T}}}}\mathbf{F}) = \mathbf{P} : \dot{\mathbf{F}} = \mathbf{P} : \mathbf{F}^{-\mathrm{T}}\dot{\overline{\mathbf{F}^{\mathrm{T}}}}\mathbf{F}. \tag{32}$$

By recalling the well-known relation $\mathbf{P}\mathbf{F}^{\mathrm{T}} = \mathbf{F}\mathbf{P}^{\mathrm{T}}$, which is a consequence of the symmetry of $\mathbf{S}$ (or $\boldsymbol{\tau}$), we find the important property $\mathbf{P} = \mathbf{F}\mathbf{P}^{\mathrm{T}}\mathbf{F}^{-\mathrm{T}}$. Note that this relationship does not directly follow from Eq. $(32)_3$, i.e. $\mathbf{P} : \dot{\mathbf{F}} = \mathbf{F}\mathbf{P}^{\mathrm{T}}\mathbf{F}^{-\mathrm{T}} : \dot{\mathbf{F}}$. Interestingly, recalling the form $\mathbf{P} : \dot{\mathbf{F}} = \mathbf{P} : \mathbf{F}^{-\mathrm{T}}\dot{\overline{\mathbf{F}^{\mathrm{T}}}}\mathbf{F}$, an equivalent relationship for $\dot{\mathbf{F}}$ does not exist, namely $\dot{\mathbf{F}} \neq \mathbf{F}^{-\mathrm{T}}\dot{\overline{\mathbf{F}^{\mathrm{T}}}}\mathbf{F}$. It is the property of $\mathbf{P}$ that we can use the equivalent forms $\dot{\Psi} = \mathbf{P} : \pounds_{\mathbf{v}}\mathbf{H} = \mathbf{P} : (\dot{\mathbf{F}} + \mathbf{F}^{-\mathrm{T}}\dot{\overline{\mathbf{F}^{\mathrm{T}}}}\mathbf{F})/2 = \mathbf{P} : \dot{\mathbf{F}}$ of the stress power.

It is the symmetry of $\mathbf{S}$ and $\boldsymbol{\tau}$ that we use $\dot{\Psi} = \mathbf{S} : \dot{\mathbf{E}} = \mathbf{S} : (\dot{\overline{\mathbf{F}^{\mathrm{T}}}}\mathbf{F} + \mathbf{F}^{\mathrm{T}}\dot{\mathbf{F}})/2$, and $\dot{\Psi} = \boldsymbol{\tau} : \pounds_{\mathbf{v}}\mathbf{e} = \boldsymbol{\tau} : (\mathbf{l} + \mathbf{l}^{\mathrm{T}})/2$ instead of $\dot{\Psi} = \mathbf{S} : \dot{\overline{\mathbf{F}^{\mathrm{T}}}}\mathbf{F}$ and $\dot{\Psi} = \boldsymbol{\tau} : \mathbf{l}$, respectively. However, it is the property of $\mathbf{P}$ that we use $\dot{\Psi} = \mathbf{P} : \dot{\mathbf{F}}$ instead of $\dot{\Psi} = \mathbf{P} : \pounds_{\mathbf{v}}\mathbf{H} = \mathbf{P} : (\dot{\mathbf{F}} + \mathbf{F}^{-\mathrm{T}}\dot{\overline{\mathbf{F}^{\mathrm{T}}}}\mathbf{F})/2$, although, by analogy with the structure above $\dot{\Psi} = \mathbf{P} : \pounds_{\mathbf{v}}\mathbf{H}$ would be the expression to use.

5. The principle of virtual work and its linearization in terms of the two-point tensor H

In this section we provide the principle of virtual work and its linearization, which constitute a fundamental basis for an implementation in a finite element program. We follow the notation and adopt standard results presented in textbooks of nonlinear continuum mechanics (see, for example, Holzapfel, 2000).

Before proceeding it is necessary to provide the Lie derivative $\pounds_{\Delta\mathbf{u}}(\pounds_{\delta\mathbf{u}}\mathbf{H})$ of $\pounds_{\delta\mathbf{u}}\mathbf{H}$ in the direction of the vector $\Delta\mathbf{u}$. By analogy with the transformation $(9)_1$ we may write

$$\pounds_{\Delta\mathbf{u}}(\pounds_{\delta\mathbf{u}}\mathbf{H}) = \mathbf{F}^{-\mathrm{T}}D_{\Delta\mathbf{u}}\delta\mathbf{E} = \frac{1}{2}[(\mathrm{grad}\Delta\mathbf{u})^{\mathrm{T}}\mathrm{Grad}\delta\mathbf{u} + (\mathrm{grad}\delta\mathbf{u})^{\mathrm{T}}\mathrm{Grad}\Delta\mathbf{u}], \tag{33}$$

where the common definitions $\mathrm{grad}\Delta\mathbf{u} = \mathrm{Grad}\Delta\mathbf{u}\mathbf{F}^{-1}$ and $\mathrm{grad}\delta\mathbf{u} = \mathrm{Grad}\delta\mathbf{u}\mathbf{F}^{-1}$ for the spatial gradients have been used. In addition, by analogy with Eq. $(9)_1$ we may write $\pounds_{\delta\mathbf{u}}\mathbf{H} = \mathbf{F}^{-\mathrm{T}}\delta\mathbf{E}$, and we deduce from $\delta W_{\mathrm{int}} = \int_{\Omega_0}\mathbf{S} : \delta\mathbf{E}\,\mathrm{d}V$ that

$$\delta W_{\mathrm{int}} = \int_{\Omega_0}\mathbf{P} : \pounds_{\delta\mathbf{u}}\mathbf{H}\,\mathrm{d}V, \tag{34}$$

where $\pounds_{\delta\mathbf{u}}\mathbf{H}$ is given by the explicit form

$$\pounds_{\delta\mathbf{u}}\mathbf{H} = \frac{1}{2}[\mathrm{Grad}\delta\mathbf{u} + (\mathrm{grad}\delta\mathbf{u})^{\mathrm{T}}\mathbf{F}]. \tag{35}$$

Now, we determine the Lie derivative of δW_{int} in the direction of $\Delta\mathbf{u}$. Thus, by means of the product rule we find from (34) that

$$\pounds_{\Delta\mathbf{u}}\delta W_{\mathrm{int}} = \int_{\Omega_0}[\pounds_{\Delta\mathbf{u}}\mathbf{P} : \pounds_{\delta\mathbf{u}}\mathbf{H} + \mathbf{P} : \pounds_{\Delta\mathbf{u}}(\pounds_{\delta\mathbf{u}}\mathbf{H})]\mathrm{d}V. \tag{36}$$

In order to proceed we need the explicit form for $\pounds_{\Delta\mathbf{u}}\mathbf{P}$. By analogy with $(12)_1$ we introduce the transformation $\pounds_{\Delta\mathbf{u}}\mathbf{P} = \mathbf{F}D_{\Delta\mathbf{u}}\mathbf{S}$, where the linearization of the second Piola–Kirchhoff stress tensor is $D_{\Delta\mathbf{u}}\mathbf{S}(\mathbf{E}) = \mathbb{C} : D_{\Delta\mathbf{u}}\mathbf{E}$, with the material elasticity tensor $\mathbb{C}$. By analogy with $(9)_1$ we get $\pounds_{\Delta\mathbf{u}}\mathbf{H} = \mathbf{F}^{-\mathrm{T}}D_{\Delta\mathbf{u}}\mathbf{E}$, which finally leads to

$$\pounds_{\Delta\mathbf{u}}\mathbf{P} = \mathbf{F}\mathbb{C} : \mathbf{F}^{\mathrm{T}}\pounds_{\Delta\mathbf{u}}\mathbf{H}, \tag{37}$$

where $\pounds_{\Delta\mathbf{u}}\mathbf{H} = [\mathrm{Grad}\Delta\mathbf{u} + (\mathrm{grad}\Delta\mathbf{u})^{\mathrm{T}}\mathbf{F}]/2$ is according to $(35)_2$.

For convenience we proceed in index notation. The double contraction of the two tensors $\pounds_{\Delta\mathbf{u}}\mathbf{P}$ and $\pounds_{\delta\mathbf{u}}\mathbf{H}$ gives the scalar function

$$\pounds_{\Delta\mathbf{u}}\mathbf{P} : \pounds_{\delta\mathbf{u}}\mathbf{H} = \pounds_{\Delta\mathbf{u}}P_{aB}\pounds_{\delta\mathbf{u}}H_{aB} = F_{aA}C_{ABCD}F_{cC}\pounds_{\Delta\mathbf{u}}H_{cD}\pounds_{\delta\mathbf{u}}H_{aB} = \pounds_{\delta\mathbf{u}}H_{aB}\hat{c}_{aBcD}\pounds_{\Delta\mathbf{u}}H_{cD}, \tag{38}$$

where the definition $\hat{c}_{aBcD} = (\hat{\mathbb{c}})_{aBcD} = F_{aA}F_{cC}C_{ABCD}$ has been introduced. Finally, (36) may be written in index and symbolic notations

$$\pounds_{\Delta\mathbf{u}}\delta W_{\text{int}} = \int_{\Omega_0} (P_{aA}\pounds_{\Delta\mathbf{u}}(\pounds_{\delta\mathbf{u}}H_{aA}) + \pounds_{\delta\mathbf{u}}H_{aB}\hat{c}_{aBcD}\pounds_{\Delta\mathbf{u}}H_{cD})\,\mathrm{d}V, \tag{39}$$

$$\pounds_{\Delta\mathbf{u}}\delta W_{\text{int}} = \int_{\Omega_0} (\mathbf{P} : \pounds_{\Delta\mathbf{u}}(\pounds_{\delta\mathbf{u}}\mathbf{H}) + \pounds_{\delta\mathbf{u}}\mathbf{H} : \hat{\mathbb{c}} : \pounds_{\Delta\mathbf{u}}\mathbf{H})\mathrm{d}V, \tag{40}$$

which constitutes a novel form for the linearization of δW_{int}. Similarly to expressions which refer to the reference and current configurations, in Eqs. (39) and (40) there are material and geometrical contributions. Note that a similar derivation of $\pounds_{\Delta\mathbf{u}}\delta W_{\text{int}}$ in terms of $\mathbf{P}$ and $\delta\mathbf{F}$ leads only to one term (no split in two parts occurs, see, for example, Holzapfel, 2000, p. 401).

Table 1 summarizes the deformation measure $\mathbf{H}$, its Lie derivative $\pounds_{\delta\mathbf{u}}\mathbf{H}$ and the linearization of $\pounds_{\delta\mathbf{u}}\mathbf{H}$, the strain-energy function and stress power, as well as the Lie derivative of the internal virtual work and its linearization in a two-point formulation. In addition, for the sake of completeness, it shows the associated expressions in the Lagrangian and Eulerian descriptions.

As clearly seen from Table 1 the structure of the two-point formulation, in which one index of the tensor quantities describes the spatial coordinates x_a, and the other the material coordinates X_A, is the analogue of those of the Lagrangian and Eulerian descriptions.

6. Example

The goal of this example is to propose a simple Saint Venant–Kirchhoff type of a strain-energy function in terms of the proposed deformation tensor $\mathbf{H}$. Additionally, we study the stress-stretch behavior of a one dimensional rod problem and analyze the growth condition.

We consider the form

$$\Psi(\mathbf{H}) = \frac{\gamma}{2}(\text{tr}\mathbf{H})^2 + \mu\text{tr}\mathbf{H}^2 \tag{41}$$

of a strain-energy function, in which $\gamma > 0$ and $\mu > 0$ are the two constants of Lamé. The Lamé constant γ is usually denoted in the literature by the symbol λ. However, in order to avoid confusion with the stretch ratio λ we use a different symbol for it. Note that the volume ratio J does not appear explicitly in the Saint Venant–Kirchhoff model.

According to Eq. (A.2), see Appendix A, a straightforward computation gives the constitutive relation

$$\mathbf{P}(\mathbf{H}) = \frac{1}{2}[\gamma\text{tr}\mathbf{H}(\mathbf{I} + \mathbf{F}^{-\mathrm{T}}\mathbf{F}^{-\mathrm{T}}) + 2\mu(\mathbf{H}^{\mathrm{T}} + \mathbf{F}^{-\mathrm{T}}\mathbf{H}\mathbf{F}^{-\mathrm{T}})]. \tag{42}$$

Now we consider *uniform extension* of a rod (with uniform cross-section) up to the stretch ratio $\lambda = l/l_0$. Then, in the transverse directions we have an equal stretch ratios, denoted by λ_{T}. Based on this kinematic assumption, the matrix representations of the tensors $\mathbf{F}$ and $\mathbf{H}$ are given by

$$[\mathbf{F}] = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda_{\mathrm{T}} & 0 \\ 0 & 0 & \lambda_{\mathrm{T}} \end{bmatrix}, \quad [\mathbf{H}] = \frac{1}{2}\begin{bmatrix} \lambda - \lambda^{-1} & 0 & 0 \\ 0 & \lambda_{\mathrm{T}} - \lambda_{\mathrm{T}}^{-1} & 0 \\ 0 & 0 & \lambda_{\mathrm{T}} - \lambda_{\mathrm{T}}^{-1} \end{bmatrix}, \tag{43}$$

Table 1
Continuum-mechanical relations in Lagrangian, two-point and Eulerian formulations

	Lagrange formulation	Two-point formulation	Eulerian formulation
Strain (def.) measure	$\mathbf{E} = \frac{1}{2}(\mathbf{F}^{\mathrm{T}}\mathbf{F} - \mathbf{I})$	$\mathbf{H} = \frac{1}{2}(\mathbf{F} - \mathbf{F}^{-\mathrm{T}}) = \mathbf{F}^{-\mathrm{T}}\mathbf{E}$	$\mathbf{e} = \frac{1}{2}(\mathbf{I} - \mathbf{F}^{-\mathrm{T}}\mathbf{F}^{-1}) = \mathbf{F}^{-\mathrm{T}}\mathbf{E}\mathbf{F}^{-1}$
Strain-energy function	$\Psi = \Psi(\mathbf{E})$	$\Psi = \Psi(\mathbf{H})$	$\Psi = \Psi(\mathbf{e})$
Stress power	$\dot{\Psi} = \mathbf{S} : \dot{\mathbf{E}}$ with $\mathbf{S} = \dfrac{\partial \Psi(\mathbf{E})}{\partial \mathbf{E}}$	$\dot{\Psi} = \mathbf{P} : £_{\mathbf{v}}\mathbf{H}$ with $\mathbf{P} = \mathbf{F}\mathbf{S}$ $\mathbf{P} = \dfrac{1}{2}\left[\dfrac{\partial \Psi(\mathbf{H})}{\partial \mathbf{H}} + \mathbf{F}^{-\mathrm{T}}\left(\dfrac{\partial \Psi(\mathbf{H})}{\partial \mathbf{H}}\right)^{\mathrm{T}}\mathbf{F}^{-\mathrm{T}}\right]$	$\dot{\Psi} = \boldsymbol{\tau} : £_{\mathbf{v}}\mathbf{e}$ with $\boldsymbol{\tau} = \mathbf{F}\mathbf{S}\mathbf{F}^{\mathrm{T}}$ $\boldsymbol{\tau} = \dfrac{\partial \Psi(\mathbf{e})}{\partial \mathbf{e}}\mathbf{b}^{-1}$
Lie derivative of the strain (def.) measure	$\delta\mathbf{E} = \mathrm{sym}(\mathbf{F}^{\mathrm{T}}\mathrm{Grad}\delta\mathbf{u})$	$£_{\delta\mathbf{u}}\mathbf{H} = \frac{1}{2}[\mathrm{Grad}\delta\mathbf{u} + (\mathrm{grad}\delta\mathbf{u})^{\mathrm{T}}\mathbf{F}] = \mathbf{F}^{-\mathrm{T}}\delta\mathbf{E}$	$£_{\delta\mathbf{u}}\mathbf{e} = \mathrm{sym}(\mathrm{grad}\delta\mathbf{u}) = \mathbf{F}^{-\mathrm{T}}\delta\mathbf{E}\mathbf{F}^{-1}$
Linearization of the Lie derivative	$D_{\Delta\mathbf{u}}\delta\mathbf{E} = \mathrm{sym}[(\mathrm{Grad}\Delta\mathbf{u})^{\mathrm{T}}\mathrm{Grad}\delta\mathbf{u}]$	$£_{\Delta\mathbf{u}}(£_{\delta\mathbf{u}}\mathbf{H}) = \frac{1}{2}[(\mathrm{grad}\Delta\mathbf{u})^{\mathrm{T}}\mathrm{Grad}\delta\mathbf{u}$ $+(\mathrm{grad}\delta\mathbf{u})^{\mathrm{T}}\mathrm{Grad}\Delta\mathbf{u}] = \mathbf{F}^{-\mathrm{T}}D_{\Delta\mathbf{u}}\delta\mathbf{E}$	$£_{\Delta\mathbf{u}}(£_{\delta\mathbf{u}}\mathbf{e}) = \mathrm{sym}[(\mathrm{grad}\Delta\mathbf{u})^{\mathrm{T}}\mathrm{grad}\delta\mathbf{u}]$ $= \mathbf{F}^{-\mathrm{T}}D_{\Delta\mathbf{u}}\delta\mathbf{E}\mathbf{F}^{-1}$
Lie derivative of the internal virtual work	$\delta W_{\mathrm{int}} = \int_{\Omega_0} \mathbf{S} : \delta\mathbf{E}\,\mathrm{d}V$	$\delta W_{\mathrm{int}} = \int_{\Omega_0} \mathbf{P} : £_{\delta\mathbf{u}}\mathbf{H}\,\mathrm{d}V$	$\delta W_{\mathrm{int}} = \int_{\Omega_0} \boldsymbol{\tau} : £_{\delta\mathbf{u}}\mathbf{e}\,\mathrm{d}V$
Linearization of δW_{int}	$D_{\Delta\mathbf{u}}\delta W_{\mathrm{int}} = \int_{\Omega_0} (\mathbf{S} : D_{\Delta\mathbf{u}}\delta\mathbf{E}$ $+\delta\mathbf{E} : \mathbb{C} : D_{\Delta\mathbf{u}}\mathbf{E})\mathrm{d}V$	$£_{\Delta\mathbf{u}}\delta W_{\mathrm{int}} = \int_{\Omega_0} (\mathbf{P} : £_{\Delta\mathbf{u}}(£_{\delta\mathbf{u}}\mathbf{H})$ $+£_{\delta\mathbf{u}}\mathbf{H} : \hat{\mathbb{c}} : £_{\Delta\mathbf{u}}\mathbf{H})\mathrm{d}V$	$£_{\Delta\mathbf{u}}\delta W_{\mathrm{int}} = \int_{\Omega_0} (\boldsymbol{\tau} : £_{\Delta\mathbf{u}}\delta\mathbf{e}$ $+£_{\delta\mathbf{u}}\mathbf{e} : \mathbb{c} : £_{\Delta\mathbf{u}}\mathbf{e})\mathrm{d}V$
Elasticity tensor	$(\mathbb{C})_{ABCD} = \dfrac{\partial S_{AB}}{\partial E_{CD}}$	$(\hat{\mathbb{c}})_{aBcD} = F_{aA}F_{cC}C_{ABCD}$	$(\mathbb{c})_{abcd} = F_{aA}F_{bB}F_{cC}F_{dD}C_{ABCD}$

and, consequently, $\mathrm{tr}\mathbf{H} = \lambda_T - \lambda_T^{-1} + (\lambda - \lambda^{-1})/2$. From (42) we find that

$$\lambda P = \frac{1}{2}(\lambda + \lambda^{-1})\left[\gamma(\lambda_T - \lambda_T^{-1}) + \left(\mu + \frac{1}{2}\gamma\right)(\lambda - \lambda^{-1})\right], \tag{44}$$

and

$$\lambda_T P_T = \frac{1}{2}(\lambda_T + \lambda_T^{-1})\left[(\gamma + \mu)(\lambda_T - \lambda_T^{-1}) + \frac{1}{2}\gamma(\lambda - \lambda^{-1})\right], \tag{45}$$

where P is the first Piola–Kirchhoff stress along the stretch ratio λ of the rod, and P_T denotes the first Piola–Kirchhoff stress in the transverse direction. In the following we shall denote the ratio γ/μ of the two constants of Lamé by the parameter ρ (recall that $\rho = 2v/(1-2v)$, with the Poisson's ratio v). Since P_T has to be zero, from (45) we may write λ_T as a function of λ, and may use the result to get an explicit expression for P in terms of λ. Hence, we find finally from (44) that

$$P^{\star} = \frac{2 + 3\rho}{4(1+\rho)}\left(\lambda - \frac{1}{\lambda^3}\right). \tag{46}$$

where $P^{\star} = P/\mu$ characterizes the normalized and dimensionless first Piola–Kirchhoff stress along λ.

Fig. 2 shows $P^{\star}$ versus λ for $\rho = 0$, which corresponds to Poisson's ratio $v = 0$, then for $\rho = 0.25$, 1, 4, and for $\rho \to \infty$, which corresponds to $v = 0.5$, i.e. the incompressible case. As can be seen, for all ρ, the function $P^{\star} = P^{\star}(\lambda)$ is monotonic in compression and tension. Note that this is not the case for the classical Saint Venant–Kirchhoff model (frequently used for engineering structures), which is characterized by the strain-energy function $\Psi(\mathbf{E}) = \gamma(\mathrm{tr}\mathbf{E})^2/2 + \mu\mathrm{tr}\mathbf{E}^2$ (see, for example, Ciarlet, 1988, p. 155, the exercise in Holzapfel, 2000, pp. 250–251). For the classical Saint Venant–Kirchhoff model $P^{\star} = P^{\star}(\lambda)$ is not monotonic in compression, and hence it does not fulfill the growth condition. The classical Saint Venant–Kirchhoff model is therefore not suitable for large compressive strains.

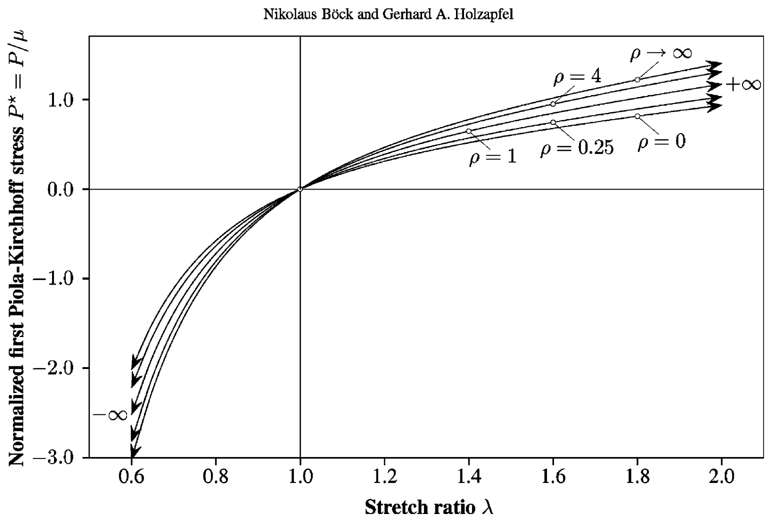

Fig. 2. Normalized first Piola–Kirchhoff stress $P^{\star} = P/\mu$ versus the stretch ratio $\lambda = l/l_0$ showing monotonic curves in compression and tension. The constitutive relation $P^{\star} = P^{\star}(\lambda)$ is derived from the strain-energy function $\Psi(\mathbf{H}) = \gamma(\mathrm{tr}\mathbf{H})^2/2 + \mu\mathrm{tr}\mathbf{H}^2$. The ratio $\rho = \gamma/\mu$ was chosen to be 0, 0.25, 1, 4 and $\to \infty$.

Note that the proposed material model satisfies the growth condition, meaning that for $\lambda \to 0^+$ the stress tends to $-\infty$, and for $\lambda \to \infty$ the stress tends to $+\infty$, which is physically realistic. Hence, this material model is also suitable for large compressive strains.

Acknowledgements

Special thanks go to Professor Ray W. Ogden, Ph.D., from the Department of Mathematics, University of Glasgow, UK, to Christian T. Gasser, Ph.D., from the Institute for Structural Analysis—Computational Biomechanics, Graz University of Technology, Austria, and to Jordi Alcoverro, Ph.D., from the Universidad Politécnica de Catalunya in Barcelona, Spain, who contributed to this paper in form of helpful discussions.

Appendix A

Constitutive equation in the two-point formulation. Here we derive the constitutive equation between the first Piola–Kirchhoff stress tensor $\mathbf{P}$ and the deformation measure $\mathbf{H}$.

Recall the definition (1) and assume a free Helmholtz-energy function according to $\Psi = \Psi(\mathbf{H})$. Thus, by means of the relations $\mathbf{l} = \dot{\mathbf{F}}\mathbf{F}^{-1}$ and $\dot{\overline{\mathbf{F}^{-\mathrm{T}}}} = -\mathbf{F}^{-\mathrm{T}}\dot{\overline{\mathbf{F}^{\mathrm{T}}}}\mathbf{F}^{-\mathrm{T}} = -\mathbf{l}^{\mathrm{T}}\mathbf{F}^{-\mathrm{T}}$ we find that

$$\dot{\Psi} = \frac{\partial \Psi(\mathbf{H})}{\partial \mathbf{H}} : \dot{\mathbf{H}} = \frac{1}{2}\frac{\partial \Psi}{\partial \mathbf{H}} : (\dot{\mathbf{F}} + \mathbf{l}^{\mathrm{T}}\mathbf{F}^{-\mathrm{T}}) = \frac{1}{2}\left[\frac{\partial \Psi}{\partial \mathbf{H}} + \mathbf{F}^{-\mathrm{T}}\left(\frac{\partial \Psi}{\partial \mathbf{H}}\right)^{\mathrm{T}}\mathbf{F}^{-\mathrm{T}}\right] : \dot{\mathbf{F}}. \tag{A.1}$$

On the other hand from $(30)_3$, and the properties $\mathbf{l} = \dot{\mathbf{F}}\mathbf{F}^{-1}$ and $\mathbf{P}\mathbf{F}^{\mathrm{T}} = \mathbf{F}\mathbf{P}^{\mathrm{T}}$ we get $\dot{\Psi} = \mathbf{P} : (\mathbf{l}\mathbf{F} + \mathbf{l}^{\mathrm{T}}\mathbf{F})/2 = \mathbf{P} : \mathbf{l}\mathbf{F}$. Consequently, for arbitrary choices of $\dot{\mathbf{F}}$, we find with $(\mathrm{A.1})_3$ the physical expression

$$\mathbf{P}(\mathbf{H}) = \frac{1}{2}\left[\frac{\partial \Psi}{\partial \mathbf{H}} + \mathbf{F}^{-\mathrm{T}}\left(\frac{\partial \Psi}{\partial \mathbf{H}}\right)^{\mathrm{T}}\mathbf{F}^{-\mathrm{T}}\right], \tag{A.2}$$

where $\mathbf{P}$ and $\mathbf{H}$ are *not* conjugate variables.

Note that general constitutive equations for incompressible hyperelastic materials may be derived from the postulate

$$\Psi = \Psi(\mathbf{H}) - p(\det\mathbf{H} - 1), \tag{A.3}$$

where the strain energy Ψ is defined for $\det\mathbf{H} = 0$ $(J = 1)$. The scalar p introduced in (A.3) serves as an indeterminate *Lagrange multiplier*.

References

Callen, H.B., 1985. Thermodynamics and an Introduction to Thermostatistics, second ed. John Wiley & Sons, New York.

Ciarlet, P.G., 1988. Mathematical Elasticity, Volume 1: Three-Dimensional Elasticity, Studies in Mathematics and its Applications, Vol. I Three–Dimensional Elasticity. Amsterdam, North-Holland.

Courant, R., Hilbert, D., 1962. Methods of Mathematical Physics. Volume II: Partial Differential Equations. John Wiley & Sons, New York.

Gyarmati, I., 1970. Non-equilibrium Thermodynamics. Springer-Verlag, Berlin/Heidelberg.

Holzapfel, G.A., 2000. Nonlinear Solid Mechanics. A Continuum Approach for Engineering. John Wiley & Sons, Chichester.

Šilhavý, M., 1997. The Mechanics and Thermodynamics of Continuous Media. Springer-Verlag, New York.